The case when A_1 is obtuse may be treated by a similar method.

Also solved by A. M. HARDING, NATHAN ALTSHILLER, C. C. YEN, J. F. LÜ, K. K. CHAN, HORACE OLSON, G. BREIT, H. C. GOSSARD, FRANK V. MORLEY, LOUIS WEISNER, and OTTO J. RAMLER.

428 (Calculus). Proposed by J. L. RILEY, Northwestern State Normal School, Tahlequah, Okla.

The loop of a lemniscate rolls in contact with the axis of x. Prove that the locus of the node is given by the equation

$$1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{a}{y}\right)^{4/3},$$

and that $2\rho\rho' = a^2$, if ρ, ρ' be corresponding radii of curvature of this locus and the lemniscate.

SOLUTION BY A. M. HARDING, University of Arkansas.

The problem, as originally stated, is incorrect. It should read as above. The equation of the lemniscate, referred to a tangent at its center, is $r^2 = a^2 \sin 2\theta$. By the formulas of the elementary calculus it is easily shown that

$$s = \int_0^\theta a\sqrt{\csc 2\theta}d\theta, \qquad \psi = 2\theta, \qquad \text{and} \qquad \rho = \frac{a}{3}\sqrt{\csc 2\theta},$$

where s is the length of arc from the node $A(x, y)$ to the point P, ψ is the angle between AP and the tangent at P, and ρ is the radius of curvature at P.

Let the lemniscate roll along the x-axis until the tangent at P coincides with this axis, and let O be the origin of coördinates. Then

$$x = OP - AP\cos\psi = a\int_0^\theta \sqrt{\csc 2\theta}d\theta - a\sqrt{\sin 2\theta}\cos 2\theta, \text{ and } y = AP\sin\psi = a\sqrt{\sin 2\theta}\sin 2\theta;$$

whence,

$$\frac{dy}{dx} = \cot 2\theta,$$

and

$$\frac{d^2y}{dx^2} = -\frac{2(\csc 2\theta)^{3/2}}{3a\sin^2 2\theta}.$$

Hence,

$$1 + \left(\frac{dy}{dx}\right)^2 = \csc^2 2\theta = \left(\frac{a}{y}\right)^{4/3}$$

and

$$\rho' = -\frac{3}{2}a\sqrt{\sin 2\theta}.$$

Hence,

$$\rho\rho' = -\frac{a^2}{2}.$$

Also solved by WILLIAM HOOVER.

430 (Calculus). Proposed by G. PAASWELL, New York City.

Revolve a circle about a chord (not a diameter). Select a system of rectilinear coördinates with this chord as one axis and the origin as the intersection of the chord and the circumference. Term this axis the z-axis and pass a plane through the x- (or y-) axis. Find the area of this surface intercepted by this plane and the xz- (or yz-) plane.

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

Let a = the radius of the circle, c = the distance of the chord from the center, 2α = the angle subtended at the center by the arc of the chord; take the middle of the chord for the origin, and the radius at right angles to the chord for the x-axis; the equation to the generating arc is then

$$z^2 + (x + a\cos\alpha)^2 = a^2,$$

or

$$x = \sqrt{a^2 - z^2} - a\cos\alpha.$$

The element of length of the arc is $ds = adz/\sqrt{a^2 - z^2}$, and the required surface

$$S = \int 2\pi x ds = 2\pi \int_0^{a\sin\alpha} (\sqrt{a^2 - z^2} - a\cos\alpha)\frac{adz}{\sqrt{a^2 - z^2}}$$

$$= 2\pi a \left[z - a\cos\alpha \int \frac{dz}{\sqrt{a^2 - z^2}} \right]_0^{a\sin\alpha}$$

$$= 2\pi a^2 \left[\sin\alpha - \cos\alpha \left(\sin^{-1}\frac{z}{a} \right)_0^{a\sin\alpha} \right] = 2\pi a^2(\sin\alpha - \alpha\cos\alpha).$$

If $\alpha = \pi/2$, $S = 2\pi a^2$, as it should be.

The object of the choice of coördinate axes as assigned in the statement of the problem is not evident.

The volume is

$$V = \pi \int x^2 dz = \pi \int_0^{a\sin\alpha} (\sqrt{a^2 - z^2} - a\cos\alpha)^2 dz$$

$$= \pi \left[a^2(1 + \cos^2\alpha)z - \tfrac{1}{3}z^3 - 2a\cos\alpha \left(\frac{z}{2}\sqrt{a^2 - z^2} + \frac{a^2}{2}\sin^{-1}\frac{z}{a} \right) \right]_0^{a\sin\alpha}$$

$$= \pi a\{\tfrac{1}{3}(2a^2 + a^2\cos^2\alpha)\sin\alpha - a\alpha\cdot a\cos\alpha\}$$

$$= \pi a\left(\frac{2a^2 + c^2}{3}\sin\alpha - a\alpha\cdot c \right), \qquad c = a\cos\alpha.$$

If $c = 0$, $V = \tfrac{2}{3}\pi a^3$, as it should be.

431 (Calculus). Proposed by J. W. LASLEY, University of North Carolina.

Explain Bertrand's fallacy:

$$\int_{x=0}^{x=1}\int_{y=0}^{y=1} \frac{x^2 - y^2}{(x^2 + y^2)^2} dydx = \int_{y=0}^{y=1}\int_{x=0}^{x=1} \frac{x^2 - y^2}{(x^2 + y^2)^2} dxdy,$$

$$\tfrac{1}{4}\pi = -\tfrac{1}{4}\pi,\ 1 = -1.$$

Solution by Horace Olson, Chicago, Illinois.

There is a theorem that the order of integration of a double integral may be reversed *if the integrand is continuous in both variables within the region of integration.* The integrand in this fallacy is discontinuous at ($x = 0$, $y = 0$). Therefore, the theorem does not justify the reversal of the order of integration.

341 (Mechanics). Proposed by PAUL CAPRON, U. S. Naval Academy.

A pole l feet long, with one end on the ground, touches the top of a wall a feet high and slides in a vertical plane perpendicular to the wall. Show that its instantaneous center of rotation is at the intersection of the vertical where it touches the ground with the perpendicular to its axis where it touches the wall, and that the locus of this center is a parabola having the latus rectum a.

Solution by S. W. Reaves, University of Oklahoma.

Let T be the point at the top of the wall and G the point on the ground through which the pole passes at some given instant. Let O be the point on the ground in the same vertical line with T, and let θ be the angle TGO.

The direction of motion of any point is clearly at right angles to the line joining that point to the instantaneous center of rotation. (See Ziwet, *Theoretical Mechanics*, Art. 23; Demartres, *Cours de Géométrie infinitésimale*, Art. 20.) Hence, if the direction of motion of a point be known, the instantaneous center C must lie on the normal at the point to the direction of motion.